Series: 1, 2, 4, 5, 17.

1. (1) $\int_0^{\pi} t^2 \cos nt \, dt$

$= \frac{1}{n}\int_0^{\pi} t^2 d(\sin nt)$

$= -\frac{2}{n}\int_0^{\pi} t \sin nt \, dt$

$= \frac{2}{n^2}\int_0^{\pi} t \, d(\cos nt)$

$= \frac{2\pi}{n^2}(-1)^n - \frac{2}{n^2}\int_0^{\pi} \cos nt \, dt$

$= \frac{2\pi}{n^2}(-1)^n$

$\int_0^{\pi} t \cos nt \, dt$

$= \frac{1}{n}\int_0^{\pi} t \, d(\sin nt)$

$= -\frac{1}{n}\int_0^{\pi} \sin nt \, dt$

$= \frac{1}{n^2}\cos nt \Big|_0^{\pi}$

$= \frac{((-1)^n - 1)}{n^2}$

$\Rightarrow \quad a \cdot \frac{2\pi}{n^2}(-1)^n + b \cdot \frac{((-1)^n - 1)}{n^2} = \frac{1}{n^2}, \ \forall n \in \mathbb{N}$

$\Rightarrow \ b = -1, \ a = \frac{1}{2\pi}$

(2) $2\cos kt \sin\frac{t}{2}$

$= \sin(\frac{t}{2} - kt) + \sin(\frac{t}{2} + kt)$

$= \sin\frac{2k+1}{2}t - \sin\frac{2(k-1)+1}{2}t$

$\Rightarrow \sum_{k=1}^{n} 2\cos kt \sin\frac{t}{2} = \sin\frac{2n+1}{2}t - \sin\frac{t}{2}$

$\Rightarrow \sum_{k=1}^{n} \cos kt = \frac{\sin\frac{2n+1}{2}t}{2\sin\frac{t}{2}} - \frac{1}{2}$

(3) $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \sum_{n=1}^{+\infty} \int_0^{\pi} (\frac{1}{2\pi}t^2 - t)\cos nt \, dt$

$= \lim_{n\to+\infty} \int_0^{\pi} (\frac{1}{2\pi}t^2 - t)\left(\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}} - \frac{1}{2}\right) dt$

$= \lim_{n\to+\infty} \int_0^{\pi} \frac{\frac{1}{2\pi}t^2 - t}{2\sin\frac{t}{2}} \sin(n+\frac{1}{2})t \, dt - \int_0^{\pi} (\frac{t^2}{4\pi} - \frac{t}{2}) dt$

$= \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{\pi^2}{6}$

2. 1). $\sum_{k=1}^{n}\frac{(-1)^k}{k} = -\sum_{k=1}^{n}\int_0^1 (-t)^{k-1}dt$

$= -\int_0^1 \frac{1-(-t)^{n-1}}{1+t}\,dt$

$\sum_{n=1}^{+\infty}\frac{(-1)^n}{n} = -\lim_{n\to\infty}\int_0^1 \frac{1-(-t)^{n-1}}{1+t}\,dt$

$= -\lim_{n\to+\infty}\int_0^1 \frac{1}{1+t}\,dt$

$= -\ln 2$

2). $\frac{(-1)^n}{2n+1} = (-1)^n\int_0^1 t^{2n}dt$

$= \int_0^1 (-t^2)^n dt$

$\Rightarrow \sum_{n=0}^{K}\frac{(-1)^n}{2n+1} = \int_0^1 \sum_{n=0}^{K}(-t^2)^n\,dt$

$= \int_0^1 \frac{1-(-t^2)^K}{1+t^2}\,dt$

$\Rightarrow \sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1} = \lim_{k\to+\infty}\int_0^1 \frac{1-(-t^2)^K}{1+t^2}\,dt$

$= \int_0^1 \frac{1}{1+t^2}\,dt$

$= \arctan t\,\Big|_0^1 = \frac{\pi}{4}$

4. $\ln(\cosh x)$

2). $\frac{1}{n+(-1)^n\sqrt{n}} \geq \frac{1}{2n}$

$\Rightarrow \sum_{n=1}^{+\infty}\frac{1}{n+(-1)^n\sqrt{n}} = +\infty$

4) $\frac{1}{\ln(n)\ln(\operatorname{ch} n)} = \frac{1}{\ln n\cdot\ln(\frac{e^n+e^{-n}}{2})} > \frac{1}{\ln n\cdot\ln e^n} = \frac{1}{n\cdot\ln n}$

$\Rightarrow \sum\frac{1}{\ln n\ln(\operatorname{ch} n)} = +\infty$

6) $\frac{n^2}{(n-1)!} \leq \frac{1}{(n-2)(n-3)} \quad (n\geq 6)$

$$\sum \frac{n^2}{(n-1)!} < +\infty$$

8). $\ln\left(\frac{2}{\pi}\arctan\frac{n^2+1}{n}\right) = \ln\left(1 - \frac{2}{\pi}\arctan\frac{n}{n^2+1}\right)$

$$\sim -\frac{2}{\pi}\cdot\frac{n}{n^2+1} \sim -\frac{2}{\pi}\cdot\frac{1}{n}$$

$$\sum \ln\left(\frac{2}{\pi}\arctan\frac{n^2+1}{n}\right) = -\infty$$

10). $n^{-\sqrt{2}\sin\left(\frac{\pi}{4}+\frac{1}{n}\right)} = n^{-\sqrt{2}\left(\frac{\sqrt{2}}{2}\cos\frac{1}{n}+\frac{\sqrt{2}}{2}\sin\frac{1}{n}\right)}$

$$= n^{-\cos\frac{1}{n}-\sin\frac{1}{n}}$$

$$= \frac{1}{n^{\sin\frac{1}{n}}}\cdot\frac{1}{n^{\cos\frac{1}{n}}}$$

$$> \frac{1}{n^{\frac{1}{n}}}\cdot\frac{1}{n} > \frac{1}{2n} \quad (n \text{ large enough})$$

$$\sum n^{-\sqrt{2}\sin\left(\frac{\pi}{4}+\frac{1}{n}\right)} = +\infty$$

5.

2). $\frac{(-1)^n}{n+(-1)^{n-1}} \overset{n=2k}{=} \frac{1}{2k-1}$

$\frac{(-1)^n}{n+(-1)^{n-1}} \overset{n=2k-1}{=} \frac{-1}{2k-1+1} = \frac{-1}{2k}$

$\Rightarrow \sum_{n=1}^{+\infty} \frac{(-1)^n}{n+(-1)^{n-1}} < +\infty$

$$\Rightarrow \sum_{n=1}^{2k} \frac{(-1)^n}{n+(-1)^{n-1}} = \sum_{n=1}^{k}\left(\frac{1}{2n-1} - \frac{1}{2n}\right)$$

$$= \sum_{n=1}^{2k} \frac{(-1)^{n-1}}{n}$$

$$\to \ln 2$$

4) $\left(\frac{\ln x}{x}\right)' = \frac{\frac{1}{x}\cdot x - \ln x}{x^2} = \frac{\ln\frac{e}{x}}{x^2} \Rightarrow \frac{\ln x}{x}$ is decreasing on $[e, +\infty)$

$$\Rightarrow \sum (-1)^n \frac{\ln n}{n} < +\infty$$

6)

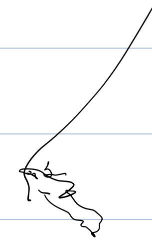
$$n!e = k + \sum_{m=n+1}^{\infty} \frac{n!}{m!}$$

$\forall m>n,\ \frac{n!}{m!} = \frac{1}{(n+1)(n+2)\cdots m} < n^{n-m}.$

$\Rightarrow \frac{1}{n+1} < \sum_{m=n+1}^{\infty} \frac{n!}{m!} < \sum_{m=n+1}^{\infty} n^{n-m} = \sum_{k=1}^{\infty} n^{-k} = \frac{\frac{1}{n}}{1-\frac{1}{n}} = \frac{1}{n-1}$

Set $e_n = \sum_{m=n+1}^{\infty} \frac{n!}{m!}$, then $e_n \in (\frac{1}{n+1}, \frac{1}{n-1})$

$K = \sum_{k=0}^{n} \frac{n!}{k!} = 1+n+$ term of even number, $\forall n \geq 1$



$\Rightarrow \left(\sin(n!\pi e)\right)^p = \left((-1)^{n-1}\sin(\pi\cdot e_n)\right)^p$

$= (-1)^{p(n-1)} \sin(\pi\cdot e_n)$

If $p \geq 2$, then $\left|\sum_n (\sin(n!\pi e))^p\right| = \left|\sum_n \sin^p(\pi\cdot e_n)\right| < \pi^p \sum_n e_n^p$

$\leq \pi^p \cdot \frac{1}{(n-1)^p} < +\infty.$

$p=1$, then $\sum_n \sin(n!\pi e) = \sum_n (-1)^{n-1} \sin(\pi\cdot e_n) < +\infty$